Bayesian Networks and Influence Diagrams: A Guide to Construction and Analysis

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Answers to Exercises

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Answer to Exercise 3.1: If the ball removed is green, then

$$\mathsf{P}(\text{blue}\,|\,\text{removed-ball-is-green}) = \frac{5-0}{10-1} = \frac{5}{9}.$$

If the ball removed is blue, then

$$\mathsf{P}(\text{blue}\,|\,\text{removed-ball-is-blue}) = \frac{5-1}{10-1} = \frac{4}{9}.$$

As the only thing known about the removed ball is that it is not red, the probability of the two events (i.e., green ball removed and blue ball removed) are identical, and being the only two possible events, the probability equals $\frac{1}{2}$. Further, as the two events are mutually exclusive, Axiom 3.3 gives us that

$$\begin{aligned}
&\mathsf{P}(\text{blue}\,|\,\text{non-red-ball-removed}) \\
&= \frac{1}{2}\cdot\mathsf{P}(\text{blue}\,|\,\text{removed-ball-is-green}) + \frac{1}{2}\cdot\mathsf{P}(\text{blue}\,|\,\text{removed-ball-is-blue}) \\
&= \left(\frac{5}{9}+\frac{4}{9}\right)\Big/2 \\
&= \frac{1}{2}.
\end{aligned}$$

Answer to Exercise 3.2:

(a)

$$\mathsf{P}(L\,|\,S) = \begin{array}{c|cc} & S=\text{true} & S=\text{false} \\ \hline L=\text{true} & \frac{1}{10} & \frac{1}{500} \\ L=\text{false} & \frac{9}{10} & \frac{499}{500} \end{array}$$

(b) Using Axiom 3.3, we get

$$\begin{aligned}
\mathsf{P}(L,S) = \mathsf{P}(L\,|\,S)\mathsf{P}(S) &= \begin{array}{c|cc} & S=\text{true} & S=\text{false} \\ \hline L=\text{true} & \frac{1}{10}\cdot\frac{1}{3} & \frac{1}{500}\cdot\frac{2}{3} \\ L=\text{false} & \frac{9}{10}\cdot\frac{1}{3} & \frac{499}{500}\cdot\frac{2}{3} \end{array} \\
&= \begin{array}{c|cc} & S=\text{true} & S=\text{false} \\ \hline L=\text{true} & \frac{1}{30} & \frac{1}{750} \\ L=\text{false} & \frac{3}{10} & \frac{499}{750} \end{array}.
\end{aligned}$$

By marginalization we get

$$\begin{aligned}
\mathsf{P}(L) &= \sum_{S}\mathsf{P}(L,S) \\
&= \left(\frac{1}{30}+\frac{1}{750}, \frac{3}{10}+\frac{499}{750}\right) \\
&= (0.0347, 0.9653).
\end{aligned}$$

Hence, we get that the population frequency of getting lung cancer is about 3.5%.

Answer to Exercise 3.3:

(a) Use Bayes' rule as follows:

   (a) Extract the vector $\mathsf{P}(\mathsf{Y} = y|\mathsf{X})$ from $\mathsf{P}(\mathsf{Y}|\mathsf{X})$.
   (b) Compute the product $\mathsf{P}(\mathsf{Y} = y|\mathsf{X})\mathsf{P}(\mathsf{X}) = \mathsf{P}(\mathsf{Y} = y, \mathsf{X})$.
   (c) Compute the normalization constant $\mu = \mathsf{P}(\mathsf{Y} = y)^{-1} = (\sum_{\mathsf{X}} \mathsf{P}(\mathsf{Y} = y, \mathsf{X}))^{-1}$.
   (d) Multiply each entry of $\mathsf{P}(\mathsf{Y} = y, \mathsf{X})$ by μ to obtain $\mathsf{P}(\mathsf{X}|\mathsf{Y} = y)$.

(b)

   (a) $\mathsf{P}(\mathsf{L} = \mathsf{true}|\mathsf{S}) = \left(\frac{1}{30}, \frac{1}{750}\right)$.
   (b) $\mathsf{P}(\mathsf{L} = \mathsf{true}, \mathsf{S}) = \mathsf{P}(\mathsf{L} = \mathsf{true}|\mathsf{S})\mathsf{P}(\mathsf{S}) = \left(\frac{1}{30} \cdot \frac{1}{10}, \frac{1}{750} \cdot \frac{1}{500}\right) = \left(\frac{1}{300}, \frac{1}{300000}\right)$.
   (c) $\mu = \mathsf{P}(\mathsf{L} = \mathsf{true})^{-1} = (\sum_{\mathsf{S}} \mathsf{P}(\mathsf{L} = \mathsf{true}, \mathsf{S}))^{-1} = \left(\frac{1}{300} + \frac{1}{300000}\right)^{-1} = 297.03$.
   (d) $\mathsf{P}(\mathsf{S}|\mathsf{L} = \mathsf{true}) = \mathsf{P}(\mathsf{L} = \mathsf{true}, \mathsf{S}) \cdot \mu = \left(\frac{1}{300}, \frac{1}{300000}\right) \cdot 297.03 = (0.9901, 0.0099)$.

   Note that the normalization constant expresses the reciprocal of the probability of the evidence ($\mathsf{L} = \mathsf{true}$ in this case).

Answer to Exercise 3.4: As the posterior probability $\mathsf{P}(\mathrm{I}|\mathsf{L})$ is proportional to the product of the prior probability $\mathsf{P}(\mathrm{I})$ and the likelihood $\mathsf{L}(\mathrm{I}|\mathsf{L}) = \mathsf{P}(\mathsf{L}|\mathrm{I})$ we get the answers as follows:

(a) $\mathsf{L}(\mathrm{I}|\mathsf{L} = \mathsf{late}) = \mathsf{P}(\mathsf{L} = \mathsf{late}|\mathrm{I}) = (0.9, 0.2)$ and thus $\mathsf{L}(\mathrm{I} = \mathsf{icy}|\mathsf{L} = \mathsf{late}) = 0.9$

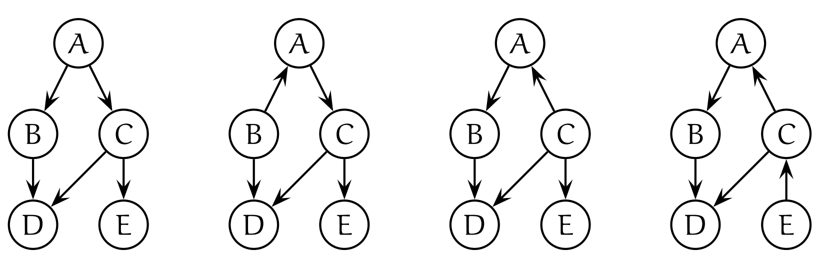
(b) Since

$$\begin{aligned} \mathsf{P}(\mathrm{I}|\mathsf{L} = \mathsf{late}) &\propto \mathsf{P}(\mathrm{I}) \times \mathsf{L}(\mathrm{I}|\mathsf{L} = \mathsf{late}) \\ &= (0.01, 0.99) \times (0.9, 0.2) \\ &= (0.009, 0.198), \end{aligned}$$

   we get (through normalization) that $\mathsf{P}(\mathrm{I}|\mathsf{L} = \mathsf{late}) = (0.04, 0.96)$, and therefore the answer is $\mathsf{P}(\mathrm{I} = \mathsf{icy}|\mathsf{L} = \mathsf{late}) = 0.04$.

Answer to Exercise 3.5:

(a) The following DAGs are equivalent independence graphs that correctly and completely represent the independence properties of P:



(b) Four DAGs fulfill the specified independence assumptions (see above).

(c) The factorizations corresponding to the four DAGs are, respectively,

$$P(A,B,C,D,E) = P(A)P(B\,|\,A)P(C\,|\,A)P(D\,|\,B,C)P(E\,|\,C)$$

$$P(A,B,C,D,E) = P(A\,|\,B)P(B)P(C\,|\,A)P(D\,|\,B,C)P(E\,|\,C)$$

$$P(A,B,C,D,E) = P(A\,|\,C)P(B\,|\,A)P(C)P(D\,|\,B,C)P(E\,|\,C)$$

$$P(A,B,C,D,E) = P(A\,|\,C)P(B\,|\,A)P(C\,|\,E)P(D\,|\,B,C)P(E)$$

Answer to Exercise 3.6:

(a)

ϕ_W:

	c_1	c_2
a_1	0.2812	0.2775
a_2	0.2871	0.1543

(b) The table is given by the vector $(1, 0)$.

(c) $\phi_U * \mathcal{E}_D$ is obtained by setting the contents of all entries of the table for ϕ_U for which $D \neq d_1$ to zero:

		c_1 d_1	c_1 d_2	c_2 d_1	c_2 d_2
a_1	b_1	0.0957	0	0.0341	0
a_1	b_2	0.1021	0	0.0634	0
a_2	b_1	0.0174	0	0.0040	0
a_2	b_2	0.0624	0	0.0307	0

(d) $\mu^{-1} = \sum_U \phi_U * \mathcal{E}_D = \phi_D(d_1) = 0.41.$